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PAIR CORRELATIONS OF SEQUENCES IN HIGHER DIMENSIONS

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ABSTRACT

We consider a system of "*generalised linear forms*" defined on a subset $\underline{x} = (x_{ij})$ of $\mathbb{R}^d$ by

$$L_1(\underline{x})(k) = \sum_{j=1}^{d_1} g_{1j}^k(x_{1j}), \ldots, L_l(\underline{x})(k) = \sum_{j=1}^{d_l} g_{lj}^k(x_{lj}) \in \mathbb{R}, \quad \text{for } k \geq 1,$$

where $d = d_1 + \cdots + d_l$ and for each pair of integers (i,j), $1 \leq i \leq l$, $1 \leq j \leq d_i$ the sequence of functions $(g_{ij}^k(\underline{x}))_{k=1}^{\infty}$ is differentiable on an interval X_{ij}. Then let

$$X_k(\underline{x}) = (\{L_1(\underline{x})(k)\}, \ldots, \{L_l(\underline{x})(k)\}) \in \mathbb{T}^l,$$

for $\underline{x}$ in the Cartesian product $\underline{X} = \times_{i=1}^{l} \times_{j=1}^{d_i} X_{ij} \subset \mathbb{R}^d$. Let $R = I_1 \times \cdots \times I_l$ be a rectangle in $\mathbb{T}^l$ and for each $N \geq 1$ let

$$V_N(R) = \sum_{1 \leq n \neq m \leq N} \chi_R(\underline{X}_n(\underline{x}) - \underline{X}_m(\underline{x}))$$

and then define

$$\Delta_N = \sup_{R \subset \mathbb{T}^l} \{V_N(R) - N(N-1)\,\mathrm{leb}(R)\}$$

where the supremum is over all rectangles in $\mathbb{T}^l$. We show that for almost every $\underline{x} \in \mathbb{T}^d$ we have that

$$\Delta_N = O(N(\log N)^{\alpha}),$$

for appropriate α. Other related results are also described.

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0. Introduction

Consider a sequence of points $\underline{y}_k = (y_k^{(1)}, \ldots, y_k^{(l)})$, $k \in \mathbb{N}$, on the l-torus $\mathbb{T}^l = [0,1)^l$. We say that this sequence is **uniformly distributed** if for any rectangle $R = I_1 \times \cdots \times I_l \subset \mathbb{T}^l$ we have that

$$\frac{\operatorname{Card}\{1 \leq k \leq N : \underline{y}_k \in R\}}{N} \to \operatorname{leb}(R), \quad \text{as } N \to +\infty,$$

where $\operatorname{leb}(\cdot)$ denotes the usual l-dimensional Lebesgue measure.

Another indication of the regularity of this sequence is if it satisfies the **pair correlation** property. The study of pair correlations of eigenvalues is a familiar tool in mathematical physics and other areas. In recent years, this quantity has been used to augment the usual uniform distributional description of certain natural sequences in the unit interval. More precisely, we can ask whether for each $s > 0$

(0.1)
$$\frac{\operatorname{Card}\{1 \leq n \neq m \leq N : ||\underline{y}_n - \underline{y}_m|| \leq sN^{-\alpha}\}}{(N-1)N^{1-l\alpha}} \to \operatorname{Vol}(B(0,s)), \quad \text{as } N \to +\infty,$$

for some fixed $\alpha > 0$, where $||\cdot||$ is the usual euclidean norm on $\mathbb{R}^l$ and $\operatorname{Vol}(B(0,s))$ is the volume of the ball of radius s (i.e., on average the $sN^{-\alpha}$ balls around points in the sequence typically contain $N(N-1)/\operatorname{Vol}(N^{-l\alpha})$ other points).

This property has been extensively studied in the particular case $l = 1$. In particular, Rudnick and Sarnak considered the sequence $x_n = \{n^k x\}$, $n \in \mathbb{N}$, and showed that for almost every $x \in [0,1)$ property (0.1) holds when $\alpha = 1$.* Rudnick and Zaharescu extended this result to sequences $(a_k x)$, where a_k, $k \geq 1$, is a lacunary sequence. Finally, Berkes, Philipp and Tichy considered analogous questions for more general sequences a_k (cf. [BPT, Proposition 4]).

In this note we shall present an extention of some of these results to higher dimensions and more general classes of functions. This is formulated in terms of general families similar to linear forms. More precisely, given a partition $d = d_1 + \cdots + d_l$ we can naturally relabel the coordinates

$$\underline{x} = (x_{11}, \ldots, x_{1d_1}, x_{21}, \ldots, x_{2d_2}, \ldots, x_{l1}, \ldots, x_{ld_l}) \in \mathbb{R}^d.$$

* In fact, Rudnick and Sarnak showed a more precise result where α is diophantine type.

We can now consider a system of "*generalised linear forms*"

$$L_1(x_{11}, \ldots, x_{1d_1})(k) = \sum_{j=1}^{d_1} g_{1j}^k(x_{1j}),$$

$$\vdots$$

$$L_l(x_{l1}, \ldots, x_{ld_l})(k) = \sum_{j=1}^{d_1} g_{lj}^k(x_{lj}),$$

where for each pair of integers (i, j) with $1 \leq i \leq l$ and $1 \leq j \leq d_i$ the sequence of functions $(g_{ij}^k(x_{ij}))_{k=1}^{\infty}$ is differentiable on the interval X_{ij}. Then let

$$X_k(\underline{x}) = (\{L_1(\underline{x})(k)\}, \ldots, \{L_l(\underline{x})(k)\}) \in \mathbb{T}^l, \tag{0.2}$$

where $\{\cdot\}$ denotes the fractional part, for $\underline{x}$ in the Cartesian product $\underline{X}$.

Example ($l = 2$, $d_1 = d_2 = 2$): For sequences of integers $a_k^{(1)}, \ldots, a_k^{(4)}$, $k \geq 1$, and $\underline{x} = (x_1, \ldots, x_4) \in \mathbb{R}^4$ we could consider the sequence of points $X_k(\underline{x}) = (\{a_k^{(1)}x_1 + a_k^{(2)}x_2\}, \{a_k^{(3)}x_3 + a_k^{(4)}x_4\}) \in \mathbb{T}^2$.

Let $R = I_1 \times \cdots \times I_l$ be a rectangle in $\mathbb{T}^l$ and let us denote for each $N \geq 1$,

$$V_N(R) = \sum_{1 \leq n < m \leq N} \chi_R(X_n(\underline{x}) - X_m(\underline{x})),$$

the counting function for pairs whose differences lie in R. We then define a measure of the discrepancy from the average

$$\Delta_N = \sup_{R \subset \mathbb{T}^l} \{V_N(R) - N(N-1)\,\mathrm{leb}(R)\}$$

where the supremum is over all rectangles in $\mathbb{T}^l$. Clearly, $|\Delta_N| \leq N(N-1)$.

Definition: We define a class of differentiable functions $(g^k(x))_{k=1}^{\infty}$ defined on the interval $X = [a, b]$ which we call of **Koksma type (K type)** if

(i) $x \mapsto g^{m\prime}(x) - g^{n\prime}(x)$, is monotone on X for $m \neq n$, and

(ii) $|g^{m\prime}(x) - g^{n\prime}(x)| \geq K > 0$ for all x in X.

We call K the Koksma constant of the family $(g^k(x))_{k=1}^{\infty}$.

Examples: The following are simple examples of functions of K type.

(a) $g^k(x) = a_k x$ for any sequence of distinct integers $(a_k)_{k=1}^{\infty}$ defined on $\mathbb{T}^1$;

(b) $g^k(x) = x^k$ $(k = 1, 2, \ldots)$, on any finite interval on $(1, \infty)$; and

(c) $g^k(x) = k^x$ $(k = 2, 3, \ldots)$, on any finite interval on $(1, \infty)$.

Definition: We say the sequence $(X_k(\cdot))_{k=1}^{\infty}$ defined by (0.2) is of **Type 1** if for each (i,j), $1 \le i \le k, 1 \le j \le d_i$ the sequence of functions $(g_{ij}^k(x))_{k=1}^{\infty}$ is of K type on the interval $X(i,j)$ with Koksma constant $K_{i,j}$.

THEOREM 1: *Suppose $(X_k(\cdot))_{k=1}^{\infty}$ is of Type 1 and let $\epsilon > 0$. Then for almost every $\underline{x} \in \mathbb{R}^d$ we have that*

$$\Delta_N = o(N(\log N)^{3+l+\epsilon}).$$

In particular, for any $\alpha < 1/l$, we can choose $R(s/N) = [-sN^{-\alpha}, sN^{-\alpha}]^l$; then we have the bound

$$\left| \frac{\mathrm{Card}\{1 \le n \ne m \le N : ||\underline{y}_n - \underline{y}_m|| \le sN^{-\alpha}\} - \mathrm{Vol}(B(0, sN^{-\alpha}))}{N^{2-l\alpha}} \right|$$
$$\le \frac{\Delta_N}{N^{2-l\alpha}} \to 0,$$

as $N \to +\infty$. In fact, the result is not necessarily true at this level of generality with $\alpha = 1/l$ (as we shall see in section 5), but on the other hand our result holds in considerable generality.

We now examine a special case.

Definition: Consider the increasing sequences of integers $(a_{ij}(n))_{n=1}^{\infty}$, for $1 \le i \le l$ and $1 \le j \le d_i$. Suppose also that for distinct pairs (i,j) and (i',j')

$$\#\{a_{ij}(n) : n \ge 1\} \cap \{a_{i'j'}(n) : n \ge 1\} < \infty. \tag{0.3}$$

We associate sequences of linear forms $L^i(\cdot)(k) : \mathbb{T}^{d_i} \to \mathbb{R}$, for $i = 1, \ldots, l$, defined by

$$L^i(\underline{x})(k) = a_{i1}(k)x_{i1} + \cdots + a_{id_i}(k)x_{id_i}.$$

We again write $\underline{X}_k = (\{L_1(\underline{x})(k)\}, \ldots, \{L_l(\underline{x})(k)\}) \in \mathbb{T}^l$. We say that $(X_k)_{k=1}^{\infty}$ is of **Type 2** if (0.3) holds.

Example: Consider a positive $d \times d$ matrix $A > 0$ with integer entries. Let $A^n = (a_{ij}(n))_{i,j=1}^d$ be the nth power. In particular, each entry $a_{ij}(n) \in \mathbb{Z}^+$ is a strictly increasing sequence. Given a typical $l \times d$ matrix $\underline{x} = (x_{ij})_{i=1}^{l}{}_{j=1}^{d}$ with entries in $\mathbb{T}$ we associate a sequence $\underline{X}_k(\underline{x}) = (\{L_1(\underline{x})(k)\}, \ldots, \{L_l(\underline{x})(k)\}) \in \mathbb{T}^l$, $k \in \mathbb{N}$, by

$$L_i(\underline{x})(k) = \sum_{j=1}^{d} a_{ij}(k)x_{ij}.$$

Let $D = ld$. For almost every $\underline{x} \in \mathbb{T}^D$ the identity (0.1) holds for the sequence $\underline{X}_k(\underline{x})$, $k \in \mathbb{N}$.

THEOREM 2: *Suppose $(X_k(\cdot))_{k=1}^{\infty}$ is of Type 2 and let $\epsilon > 0$. Then for almost every $\underline{x} \in \mathbb{R}^d$ we have that*

$$\Delta_N = o(N(\log N)^{1+l+\epsilon}).$$

When $l = 1$, Theorem 2 reduces to a result in [BTP].

In the case $d = 1$ this can be strengthened under an additional hypothesis. Suppose instead of being a sequence of distinct integers as in Theorem 2 we assume that $(a_n)_{n=1}^{\infty}$ is a sequence of distinct reals such that for some real $q > 0$ we have $|a_k - a_j| \geq q$ for each pair of distinct natural numbers j and k. Then we say $(X_k(\cdot))_{k=1}^{\infty}$ is of **Type 3**.

THEOREM 3: *Suppose $(X_k(\cdot))_{k=1}^{\infty}$ is of Type 3 and let $\epsilon > 0$. Then for almost every $\underline{x} \in \mathbb{R}$ we have that*

$$\Delta_N = o(N(\log N)^{2+\epsilon}).$$

When $l = 1$, Theorem 3 reduces to a result in [BTP] when a_k is integer.

It is unknown to the authors if Theorem 3 can be extended to higher dimensions. In section 4 we will present an example which shows Theorems 1, 2 and 3 are, in some sense, near to best possible. Moreover, this example also shows that property (0.1) established by Rudnick and Sarnak for sequences like $a_n = n^k$ $(n = 1, 2, \cdots)$ with integer $k \geq 1$ does not extend to all strictly increasing sequences of integers $(a_n)_{n=1}^{\infty}$.

For each pair (i, j) with $1 \leq i \leq k, 1 \leq j \leq d_i$ let $(a_{ij}^n)_{n=1}^{\infty}$ be a strictly increasing sequence of integers. We can let

$$g_{ij}(x)(k) = a_{ij}^k \cos(a_{ij}^k x).$$

Also write

$$L_i(\underline{x})(k) = \sum_{j=1}^{d_i} g_{ij}(x_{ij})(k) \quad (1 \leq i \leq l)$$

and let $X_k(\underline{x}) = (\{L_1(\underline{x})(k)\}, \ldots, \{L_l(\underline{x})(k)\})$. Then we say $(X_k(\cdot))_{k=1}^{\infty}$ is of **Type 4**. Our final theorem is the following

THEOREM 4: *Suppose $(X_k(\cdot))_{k=1}^{\infty}$ is of Type 4 and let $\epsilon > 0$. Then for almost every x we have $\Delta_N(x) = o(N^{3/2}(\log N)^{l+3+\epsilon})$.*

Because its formulation is slightly more straightforward we shall prove Theorem 2 in section 1 before we prove Theorem 1 in section 2. Because of its similarity to the proof of Theorem 2, the proof of Theorem 3 is only sketched in section 1.

1. Proof of Theorem 2

We begin with some standard notation. Let $R = I_1 \times \cdots \times I_l$ be a rectangle in $\mathbb{T}^l$ (where $I_1, \ldots, I_l \subset \mathbb{T}$ are intervals). Given $\underline{y}_1, \ldots, \underline{y}_N \in \mathbb{T}^l$ we can denote

$$D_N = \sup_{R \subset \mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \chi_R(y_j) - \mathrm{leb}(R) \right|.$$

We denote $e(\cdot) = \exp(2\pi i \cdot)$ and let $\langle ., . \rangle$ denote the standard inner product on $\mathbb{R}^l$. The following result is standard (cf. [KN, p. 116]). Here and henceforth C, possibly with subscripts, denotes a positive constant not necessarily the same at each occurence.

Lemma 1.1 (Erdős–Turan–Koksma–Szüsz inequality): *There exists $C_1 > 0$ such that for any $L \geq 1$, and $\underline{y}_1, \ldots, \underline{y}_M \in \mathbb{T}^l$,*

$$M D_M \leq C_1 \left(\frac{M}{L} + \sum_{\substack{0 < M(h) \leq L \\ \underline{h} \in \mathbb{Z}^l}} \frac{1}{r(\underline{h})} \left| \sum_{1 \leq i \leq M} e(\langle \underline{h}, \underline{y}_i \rangle) \right| \right)$$

where $\underline{h} = (h_1, \ldots, h_l) \in \mathbb{Z}^l$ and we denote

$$r(\underline{h}) = \prod_{i=1}^{l} \max(1, |h_i|) \quad \textit{and} \quad M(\underline{h}) = \max_{1 \leq i \leq l} |h_i|.$$

We want to apply Lemma 1.1 with the M terms $\underline{y}_j$ replaced by $N(N-1)$ terms $\underline{X}_i(\underline{x}) - \underline{X}_j(\underline{x})$ (with $1 \leq i < j \leq N$) to deduce that

$$\begin{aligned} \Delta_N &\leq C_1 \left(\frac{N^2}{L} + \sum_{\substack{0 < M(h) \leq L \\ \underline{h} \in \mathbb{Z}^l}} \frac{1}{r(\underline{h})} \left| \sum_{1 \leq j \neq k \leq N} e(\langle \underline{h}, \underline{X}_j(\underline{x}) - \underline{X}_k(\underline{x}) \rangle| \right) \\ &\leq C_1 \left(\frac{N^2}{L} + \sum_{\substack{0 < M(h) \leq L \\ \underline{h} \in \mathbb{Z}^l}} \frac{1}{r(\underline{h})} \left(\left| \sum_{1 \leq j \leq N} e(\langle \underline{h}, \underline{X}_j(\underline{x}) \rangle) \right|^2 + N \right) \right), \end{aligned} \tag{1.1}$$

the last line coming from the trivial identity

$$N + \sum_{1 \leq j \neq k \leq N} e(\langle \underline{h}, \underline{X}_j(\underline{x}) \rangle) e(-\langle \underline{h}, \underline{X}_k(\underline{x}) \rangle) = \left| \sum_{1 \leq j \leq N} e(\langle \underline{h}, \underline{X}_j(\underline{x}) \rangle) \right|^2.$$

If we set $L = 2^K$, say, for some $K \geq 1$, we can use (1.1) to get the estimate

(1.2)

$$\int_{\mathbb{T}^d} \Big| \max_{2^{K-1} \leq m < 2^K} \Delta_m \Big| d\underline{x}$$

$$\leq C_1 \Bigg(2^K + \sum_{\substack{0 < M(h) \leq 2^K \\ \underline{h} \in \mathbb{Z}^l}} \frac{1}{r(\underline{h})} \int_{\mathbb{T}^d} \Bigg[\max_{2^{K-1} \leq r < 2^K} \Bigg| \sum_{1 \leq j \leq r} e(\langle \underline{h}, \underline{X}_j(\underline{x}) \rangle) \Bigg|^2 d\underline{x} + 2^K \Bigg] \Bigg).$$

To proceed we need the following estimate.

Lemma 1.2: *There exists $C_2 > 0$ such that for any $N \geq 1$,*

$$\int_{\mathbb{T}^d} \Bigg(\max_{1 \leq n \leq N} \Bigg| \sum_{1 \leq j \leq n} e(\langle \underline{h}, \underline{X}_j(\underline{x}) \rangle) \Bigg|^2 \Bigg) d\underline{x} \leq C_2 N, \tag{1.3}$$

uniformly in $\underline{h} = (h_1, \ldots, h_l) \in \mathbb{Z}^l$

Proof: This follows from a result of Fefferman and Sjölin (cf. [Na1, Cor. 12]), which is in turn a multidimensional version of the Carleson–Hunt maximal inequality [2]. Their theorem states that there exists $C_3 > 0$ such that whenever $f \in L^2(\mathbb{T}^d)$ has a Fourier expansion $f(\underline{x}) = \sum_{\underline{n} \in \mathbb{Z}^d} a_{\underline{n}} e(\langle \underline{n}, \underline{x} \rangle)$ and if we write $\underline{n} = (n_1, \ldots, n_k)$ and denote by $S_\kappa f(\underline{x}) = \sum_{|n_1|, \ldots, |n_d| \leq \kappa} a_{\underline{n}} e(\langle \underline{n}, \underline{x} \rangle)$ the truncation to a $(2\kappa + 1)$-square, then we have the bound

$$\int_{\mathbb{T}^d} \max_{\kappa > 0} |S_\kappa f(\underline{x})|^2 d\underline{x} \leq C_3 \int_{\mathbb{T}^d} |f(\underline{x})|^2 d\underline{x}. \tag{1.4}$$

We want to apply (1.4) to the function

$$f(\underline{x}) = \sum_{j=1}^{N} e(\langle \underline{h} X_j(\underline{x}) \rangle) = \sum_{j=1}^{N} e\Bigg(\sum_{i=1}^{k} h_i L_i(\underline{x})(j) \Bigg) = \sum_{j=1}^{N} e\Bigg(\sum_{i=1}^{k} \sum_{l=1}^{d_i} h_i a_{il}(j) x_{il} \Bigg).$$

However, because of (0.3), there is little scope for cancellation in the exponential and so $N + O(1)$ Fourier coefficients are equal to 1 and all of the others are zero. ∎

Applying (1.3) with $N = 2^K$ gives a bound on (1.2) of

$$\int_{\mathbb{T}^d} \Big| \max_{2^{K-1} \leq m < 2^K} \Delta_m \Big| d\underline{x} \leq 2^K \Bigg(C_1 + (C_2 + 1) \sum_{\substack{0 < M(h) \leq 2^K \\ \underline{h} \in \mathbb{Z}^l}} \frac{1}{r(\underline{h})} \Bigg). \tag{1.5}$$

To bound the term on the right hand side of (1.5) we need the following lemma.

LEMMA 1.3: *There exists $C_3 > 0$ such that for any $L \geq 2$,*

$$\sum_{\substack{0<M(h)\leq L \\ \underline{h}\in\mathbb{Z}^l}} \frac{1}{r(\underline{h})} \leq C_3(\log L)^l.$$

Proof: This follows from a combinatorial argument in [Na1]. Alternatively, this follows directly by bounding the summation by the product of integrals

$$\left(2 + 4\int_1^L \frac{dx}{x}\right)^l = (2 + 4\log L)^l. \qquad \blacksquare$$

Using Lemma 1.3 with $L = 2^K$ to bound the right hand side of (1.5) gives

$$\text{(1.6)} \quad \int_{\mathbb{T}^d} |\max_{2^{K-1}\leq m<2^K} \Delta_m| d\underline{x} \leq 2^K(C_1 + (C_2+1)C_3(\log 2)^l K^l) \leq C_4 2^K K^l,$$

for suitably large $C_4 > 0$. Given $\epsilon > 0$, we define

$$E_\epsilon = \{\underline{x} \in \mathbb{T}^d : \Delta_N(\underline{x}) > N(\log N)^{k+1+\epsilon} \text{ for infinitely many } N\}.$$

We are now in a position to prove the following result.

LEMMA 1.4: *For any $\epsilon > 0$, the set E_ϵ has zero Lebesgue measure.*

Proof: If we denote, for each $K \geq 1$,

$$A_K = \{\underline{x} \in \mathbb{T}^d : |\max_{2^{K-1}\leq m<2^K} \Delta_m(\underline{x})| > 2^K K^{l+1+\epsilon}\},$$

then one easily sees $E_\epsilon \subset \bigcap_{r=1}^\infty \bigcup_{K=r}^\infty A_K$. Using (1.6) we can bound

$$\begin{aligned} \mathrm{leb}(A_K) &\leq \frac{\int_{\mathbb{T}^d} |\max_{2^{K-1}\leq m<2^K} \Delta_m| d\underline{x}}{2^K K^{l+1+\epsilon}} \\ &\leq \frac{C_4 2^K K^l}{K^{l+1+\epsilon} 2^K} \leq C_5 K^{-(1+\epsilon)}, \end{aligned}$$

for sufficiently large $C_5 > 0$. In particular, we can now observe that

$$\sum_{K=1}^\infty \mathrm{leb}(A_K) \leq \sum_{K=1}^\infty K^{-(1+\epsilon)} < +\infty.$$

It follows from the Borel-Cantelli lemma that $\mathrm{leb}(E_\epsilon) = 0$. $\blacksquare$

In particular, Theorem 2 now follows from Lemma 1.4 above.

A similar argument can be used to prove Theorem 3. The only additional ingredient is the estimate given in the following Lemma, taken from [MV] which both replaces and generalises the role played by the Carleson–Hunt maximal inequality in the proof of Theorem 2 for $d = 1$.

Lemma 1.5: *Suppose we are given $q > 0$, real numbers $(a_n)_{n=1}^N$ such that $a_{n+1} - a_n \geq q$, real numbers T and T_0 with $T > 0$ and complex numbers $(b_n)_{n=1}^N$. Then there exists $C > 0$ such that*

$$\int_{T_0}^{T_0+T} \left(\max_{1 \leq \nu \leq N} \left| \sum_{n=1}^{\nu} b_n e(a_n t) \right|^2 \right) dt \leq C\Big(T + \frac{2\pi}{q}\Big) \sum_{n=1}^{N} |a_n|^2.$$

2. Koksma sequences

For background to some of the ideas used in this section we refer the reader to [C] and [Na1]. Let $(Y_t)_{t=1}^{\infty}$ be a sequence of measurable functions defined on a measure space Ω and then write

$$S_j = \sum_{1 \leq t \leq j} Y_t, \quad \text{for } j = 1, 2, \ldots .$$

We can define

$$Y_{rs} = \sum_{r \leq t < s} Y_t \quad (= S_s - S_r), \qquad \text{for } r < s,$$

and let $M_n = \sup_{1 \leq j \leq n} |S_j|$. We have the following elementary lemma.

Lemma 2.1: *For $K \geq 1$,*

$$\int_{\Omega} M_{2^K}(\omega) d\omega \leq (K+1) \Bigg(\sum_{i=1}^{K+1} \sum_{\nu=1}^{2^i - 1} \int_{\Omega} |Y_{\nu 2^{(K+1)-i}, (\nu+1) 2^{(K+1)-i}}|^2(\omega) d\omega \Bigg).$$

Proof: Suppose $j < 2^{K+1}$. We can find $h < K+1$ and write

$$S_j = \sum_{i=1}^{h} Y_{l_i l_{i+1}}, \tag{2.1}$$

where $l_i < l_{i+1}$, with $l_1 = 0$ and $l_i \in \{\nu 2^{(K+1)-i}\}_{\nu=0}^{2^i-1}$, for $i = 2, \ldots, h-1$, and $l_h = j$. Applying the triangle inequality to (2.1) we can write

$$|S_j| \leq \sum_{i=1}^{h} |Y_{l_i l_{i+1}}| \quad \text{for } j = 1, \ldots, h. \tag{2.2}$$

Since the terms $|Y_{l_i, l_{i+1}}|$ are positive we can apply the Cauchy–Schwartz inequality to bound

$$|S_j|^2 \leq (K+1) \sum_{i=1}^{h} |Y_{l_i, l_{i+1}}|^2 \quad \text{for } j = 1, \ldots, h.$$

Taking a supremum over j implies for all $n \le 2^K$ that

$$M_n^2 = (\sup_{1\le i\le n} |S_i|)^2 = \sup_{1\le i\le n} |S_i|^2$$
$$\le (K+1)\Bigg(\sum_{i=1}^{K+1} \sum_{\nu=1}^{2^i-1} |Y_{\nu 2^{(K+1)-i},(\nu+1)2^{(K+1)-i}}|^2 \Bigg).$$

Integrating both sides over Ω completes the proof. ∎

LEMMA 2.2: *Let $\{g_1, \dots, g_N\}$ denote a finite K type defined on the interval X with Koksma constant K. If we denote*

$$d^*_{m,n} = \inf_{x\in X} |g'_m(x) - g'_n(x)| \quad (m \ne n),$$

then we have

$$\sum_{1\le j<\le k\le N} (d^*_{k,j})^{-1} \le K^{-1} \log(3N).$$

Proof: By hypothesis, for each pair of distinct natural numbers j, k the function $g'_k(x) - g'_j(x)$ is monotone. In particular, we can find a permutation $\pi(1), \dots, \pi(N)$ of the set $\{1, \dots, N\}$ such that if $k > j$,

$$g'_{\pi(k)}(x) \ge g'_{\pi(j)}(x),$$

for all $x \in X$. Thus, either

$$g'_{\pi(k)}(x) \ge g'_{\pi(j)}(x) = \sum_{i=j}^{k-1} (g_{\pi(i+1)})'(x) - (g_{\pi(i)})'(x) \ge (k-j)K$$

or

$$d_{\pi(k),\pi(j)} \ge (k-j)K.$$

In particular,

$$\sum_{1\le j<k\le N} (d^*_{k,j})^{-1} = \sum_{1\le j<k\le N} (d^*_{\pi(k),\pi(j)})^{-1}$$
$$\le K^{-1} \sum_{1\le j<k\le N} (k-j)^{-1}$$
$$\le K^{-1} N \log(3N). \quad ∎$$

LEMMA 2.3: *Suppose $(g_k(x))_{k=1}^{\infty}$ defined on the interval $X = [a, b]$ is of K type. For $m \ne n$ set*

$$d^*_{m,n} = \inf_{x\in X} |g'_m(x) - g'_n(x)|.$$

Then for $h \neq 0$, there exists $C > 0$ such that

$$\left|\int_a^b e((g_m(x) - g_n(x))h)dx\right| \leq \frac{C}{hd^*_{m,n}}.$$

Proof: Using the second mean value theorem there exist x_0 and x_0' in $[a, b]$ such that

$$\begin{aligned}&\left|\int_a^b e((g_m(x) - g_n(x))h)dx\right| = \left|\frac{1}{2\pi i}\int_a^b \frac{d(e(g_m(x) - g_n(x))h)}{g_m'(x) - g_n'(x)}dx\right|\\ &\leq\left|\frac{1}{2\pi h}\left(\int_a^{x_0} \frac{d(\sin(g_m(x) - g_n(x)h))}{g_m'(a) - g_n'(a)}dx + \int_{x_0}^b \frac{d(\sin(g_m(x) - g_n(x)h))}{g_m'(b) - g_n'(b)}\right)dx\right|\\ &\quad+\left|\frac{1}{2\pi h}\left(\int_a^{x_0'} \frac{d(\cos(g_m(x) - g_n(x)h))}{g_m'(a) - g_n'(a)}dx + \int_{x_0'}^b \frac{d(\cos(g_m(x) - g_n(x)h))}{g_m'(b) - g_n'(b)}\right)dx\right|.\end{aligned}$$

We can bound the last line by

$$\frac{1}{\pi h}\Big(\frac{2}{|g_m'(a) - g_n'(a)|} + \frac{2}{|g_m'(b) - g_n'(b)|}\Big). \qquad \blacksquare$$

Let $h = (h_1, \ldots, h_l)$ be an l-tuple in $\mathbb{Z}^l$ and let

$$I_h(N) = \int_X \left|\sum_{n=1}^N e(\langle X_n, h\rangle)\right|^2 dx, \quad \text{for } N \in \mathbb{N}.$$

Lemma 2.4: *For $I_h(N)$ defined above we have the bound*

$$I_h(N) << N(1 + (\log N))\left(\sum_{i:h_i\neq 0} |h_i|^{-1}\right).$$

Proof: We can expand

$$\begin{aligned}I_h(N) &= \int_X \left(\sum_{n=1}^N e(\langle X_n, h\rangle)\right)\overline{\left(\sum_{n=1}^N e(\langle X_n, h\rangle)\right)}dx\\ &= \int_X \sum_{n=1}^N\sum_{m=1}^N e(\langle X_n - X_m, h\rangle)dx\\ &= \sum_{n=1}^N\sum_{m=1}^N \int_X e(\langle X_n - X_m, h\rangle)dx.\end{aligned}$$

This means that

$$I_h(N) \leq C\left(N + 2\sum_{1\leq m<n\leq N}\left|\int_X e(\langle X_m - X_n, h\rangle)dx\right|\right).$$

By definition

$$\int_X e(\langle X_m - X_n, h\rangle)dx = \prod_{i=1}^{l}\prod_{j=1}^{d_i}\Big(\int_{X_{ij}} e((g_m(i,j,x) - g_n(i,j,x))h_i)dx_{ij}\Big).$$

Let

$$d_{m,n,i,j} = \inf_{x\in X_{ij}} |g'_m(i,j,x) - g'_n(i,j,x)|$$

and set

$$d_{m,n,i,j}* = \begin{cases} 1 & \text{if } h_j = 0, \\ h_j d_{m,n,i,j} & \text{if } h_j \neq 0. \end{cases}$$

Using Lemma 2.3 we have that

$$I_h(N) \leq C\Big(N + \sum_{1\leq m<n\leq N} \prod_{i=1}^{l}\prod_{j=1}^{d_i}\Big(\frac{1}{d^*_{m,n,i,j}}\Big)\Big).$$

Applying the arithmetic mean geometric mean inequality gives that

$$\begin{aligned} I_h(N) &\leq C\Big(N + \sum_{1\leq m<n\leq N}\Big[\frac{1}{d}\sum_{i=1}^{l}\sum_{j=1}^{d_i}\Big(\frac{1}{d^*_{m,n,i,j}}\Big)^d\Big]\Big) \\ &\leq C\Big(N + \sum_{i=1}^{l}\sum_{j=1}^{d_i}\sum_{1\leq m<n\leq N}\Big(\frac{1}{d^*_{m,n,i,j}}\Big)\Big). \end{aligned}$$

Using Lemma 2.2 this gives

$$I_h(N) \leq C\Big(N + \sum_{i:h_i\neq 0}\sum_{j=1}^{d_i}\frac{N\log(3N)}{|h_i|}\Big) \leq C\Big(N + \sum_{i:h_i\neq 0}\frac{N\log(3N)}{|h_i|}\Big),$$

as required. ∎

Lemma 2.5: *For $(X_t)_{t=1}^{\infty}$ of Type 1, we have the bound*

$$\int_X \Big(\max_{1\leq m\leq 2^{K+1}}\Big|\sum_{j=1}^{m} e(\langle X_j, h\rangle)\Big|\Big)^2 dx \leq C(K+1)^2 2^{K+1}\Big(1+(K+1)\sum_{i:h_i\neq 0}\frac{1}{|h_i|}\Big).$$

Proof: We shall apply Lemma 2.1. Let $Y_t = e(\langle X_t, h\rangle)$. Then Lemma 2.4 gives

$$\int_X |X_{\nu 2^{K+1-i},(\nu+1)2^{K+1-i}}|^2 dx \leq C\Big(2^{K+1-i} + \sum_{i:h_i\neq 0}\frac{2^{K+1-i}\log(3.2^{K+1-i})}{|h_i|}\Big).$$

So

$$\sum_{\nu=1}^{2^i-1}\int_X |X_{\nu 2^{K+1-i},(\nu+1)2^{K+1-i}}|^2 dx \leq C\Big(2^{K+1} + \log(3.2^{K+1})\sum_{i:h_i\neq 0}\frac{1}{|h_i|}\Big),$$

which proves Lemma 2.5. ∎

Using (1.2)

$$\int_X |\max_{2^{K-1}\le m<2^K} \Delta_m| d\underline{x}$$
$$\le C_1\Bigg(2^K + \sum_{\substack{0<M(h)\le 2^K\\ \underline{h}\in\mathbb{Z}^l}} \frac{1}{r(\underline{h})}\int_X \Bigg[\max_{2^{K-1}\le r<2^K}\Bigg|\sum_{1\le j\le r} e(\langle \underline{h}, \underline{X}_j\rangle)\Bigg|^2 d\underline{x} + 2^K\Bigg]\Bigg),$$

which, using Lemma 2.5, is

$$\le C\Bigg(2^K + \sum_{0<M(h)\le 2^K} \frac{1}{r(h)}(K+1)^2 2^{K+1}\Bigg(K + \sum_{i:h_i\ne 0}\frac{1}{|h_i|}\Bigg)\Bigg).$$

To proceed we need the following.

Lemma 2.6: *For $h = (h_1, \ldots, h_l) \in \mathbb{Z}^l$ and $r(h), M(h)$ defined as in Lemma 1.1, then there exists $C_3 > 0$ such that*

$$\sum_{\substack{0<M(h)\le L\\ \underline{h}\in\mathbb{Z}^l}} \frac{1}{r(\underline{h})}\Bigg(\sum_{i:h_i\ne 0}\frac{1}{|h_i|}\Bigg) \le C_3(\log L)^{l-1} \quad (L\in\mathbb{N}).$$

Proof:

$$\sum_{\substack{0<M(h)\le 2^K\\ \underline{h}\in\mathbb{Z}^l}} \frac{1}{r(\underline{h})}\Bigg(\sum_{i:h_i\ne 0}\frac{1}{|h_i|}\Bigg) \le 2^l \sum_{\substack{0<M(h)\le 2^K\\ \underline{h}\in\mathbb{W}^l/\{0\}}} \frac{1}{r(\underline{h})}\Bigg(\sum_{i:h_i\ne 0}\frac{1}{|h_i|}\Bigg),$$

where $\mathbb{W}^k = \{0, 1, 2, \ldots,\}^k$. If we denote $\{0, 1, \ldots, d\}$ by $[d]$, denote $\{\epsilon \in 2^{[d]} : \#e = l\}$ by $2^{[d]}_l$ and set

$$r'(h) = \Pi_{i=1}^l \max\{1, h_i\};$$

this is

$$\le C \sum_{\substack{0<M(h)\le 2^K\\ \underline{h}\in\mathbb{W}^l/\{0\}}} \frac{1}{r'(\underline{h})}\Bigg(\sum_{i:h_i\ne 0}\frac{1}{|h_i|}\Bigg),$$

which is

$$C\Bigg(\sum_{b=1}^{l}\sum_{\tau\in 2^{[d]}_k}\Bigg(\sum_{\substack{i_1,\ldots,i_b=1\\ \tau\in\{i_1,\ldots i_b\}}} \frac{1}{h_{i_1}\cdots h_{i_b}}\Bigg)\Bigg)\Bigg(\sum_{i:h_i\ne 0}\frac{1}{h_i}\Bigg)$$
$$\times\, C\Bigg\{\sum_{b=1}^{l} l\binom{l}{b}\Bigg(\sum_{j=1}^{L} j^{-1}\Bigg)^{b-1}\Bigg(\sum_{j=1}^{L} j^{-2}\Bigg)\Bigg\} << (\log L)^{l-1},$$

as required. Lemma 2.6 is proved. ∎

Lemma 2.6 immediately gives

$$\int_X \left(\max_{2^{K-1} \le m \le 2^K} \left| \sum_{j=1}^{l} e(< X_j, h >) \right| \right)^2 dx$$
$$\le C(2^K + (K+1)^2 2^{K+1} K^l + 2^K (K+1)^3 K^{l-1})$$
$$<< 2^K K^{l+2}.$$

The proof of Theorem 1 now follows by an argument analogous to that used to prove Lemma 1.4. ∎

3. Oscilatory sequences

Again we need a series of Lemmas. The first is a quasi-orthogonality lemma due to LeVeque [L].

LEMMA 3.1: *For integers $h(\neq 0)$, j and k ($j \neq k$) and a given interval $[u, v]$*

$$\left| \int_u^v e(h(a_j \cos a_j x - a_k \cos(a_k x)) dx \right| \le \frac{K}{|a_k - a_j|^{1/2}}.$$

We have the following essential two norm estimate.

LEMMA 3.2: *For integer $h \neq 0$ we have*

$$J_h(M, N) = \int_X \left| \sum_{n=M+1}^{M+N} e(< X_n, h >) \right|^2 dx \le C N^{3/2}.$$

Proof: Straightforwardly we have

$$|J_h(M, N)| \le C \left(N + \sum_{M+1 \le m < n \le M+N} \left| \int_X e(< X_m - X_n, h >) dx \right| \right).$$

Also

$$\left| \int_X e(< X_m - X_n, h >) dx \right|$$
$$\le \Pi_{i=1}^{l} \Pi_{j=1}^{d_k} \left| \int_{X(i,j)} e(a_m(i,j) \cos(a_m(i,j) x_{i,j}) - e(a_n(i,j) \cos(a_n(i,j) x_{i,j}) dx_{i,j} \right|.$$

Let $z^*_{m,n,i,j} = 1$ if $h_j = 0$ and $z^*_{m,n,i,j} = h_j z_{m,n,i,j}$ if $h_j \neq 0$, where

$$z_{m,n,i,j} = |a_m(i,j) - a_n(i,j)|^{1/2} \ge |m - n|^{1/2}.$$

Thus

$$|J_h(M,N)| \leq C\Big(N + \sum_{M+1\leq m<n\leq N} \Pi_{i=1}^{l}\Pi_{j=1}^{d_k}\Big(\frac{1}{z^*_{m,n,i,j}}\Big)\Big).$$

Using the arithmetic mean geometric mean inequality this is

$$|J_h(M,N)| \leq C\Big(N + \sum_{M+1\leq m<n\leq N} \sum_{i=1}^{l}\sum_{j=1}^{d_k}\Big(\frac{1}{z^*_{m,n,i,j}}\Big)\Big),$$

which is

$$\leq C\Big(N + \sum_{M+1\leq m<n\leq M+N} (n-m)^{-1/2}\Big) \leq CN^{3/2},$$

as required. ∎

We also have the following lemma.

Lemma 3.3: *We have*

$$\int_X \Big(\max_{1\leq m\leq 2^{K+1}}\Big|\sum_{j=1}^{l} e(< X_j, h >)\Big|^2\Big)dx \leq Ck2^{3K/2}.$$

Proof: Using Lemma 3.2 we see that

$$\int_X |Y_{\nu 2^{K+1-i},(\nu+1)2^{K+1-i}}|^2 dx \leq C(2^{K+1-i})^{3/2}.$$

This means that

$$\sum_{i=1}^{2^i-1}\int_X |Y_{\nu 2^{K+1-i},(\nu+1)2^{K+1-i}}|^2 dx \leq C2^{3(K+1)/2}2^{-i/2}$$

and hence that

$$(K+1)\sum_{i=1}^{K+1}\sum_{i=1}^{2^i-1}\int_X |Y_{\nu 2^{K+1-i},(\nu+1)2^{K+1-i}}|^2 dx \leq C(K+1)2^{3(k+1)/2}.$$

Invoking Lemma 2.1 completes the proof of Lemma 3.3. ∎

We can now complete the proof of Theorem 4. Recall that

$$\int_X |\max_{2^{K-1}\leq m<2^K} \Delta_m|^2 dx$$

$$\leq C\Big(2^K + \sum_{0\leq M(h)\leq 2^K} \frac{1}{r(h)} \int_X \Big(\max_{1\leq m\leq 2^{K+1}} \Big|\sum_{j=1}^{m} e(< X_j, h >)\Big|^2\Big) dx\Big)$$

$$\leq C\Big(2^K + \sum_{0\leq M(h)\leq 2^K} 1/r(h) K 2^{3K/2}\Big)$$

$$\leq C\Big(2^{3K/2} K\Big(\sum_{0\leq M(h)\leq 2^K} \frac{1}{r(h)}\Big)\Big),$$

which, using Lemma 1.3, is

$$\leq C 2^{3K/2} K^{l+1}.$$

The theorem now follows using this estimate and the argument used to complete the proof of Theorem 1. ∎

4. Sharpness of Theorems 1, 2 and 3

The following is the standard Denjoy–Koksma Theorem (for $d = 1$).

Lemma 4.1: *Let $f\colon \mathbb{T} \to \mathbb{R}$ be a continuous function of bounded variation. For any points $(x_i)_{i=1}^N \subset \mathbb{T}$ we can bound*

$$\left|\frac{1}{N}\sum_{j=1}^{N} f(x_j) - \int_0^1 f(t)dt\right| \leq 6V(f)D(x_1, \ldots, x_N),$$

where $V(f) = \sup_{(x_i)_{i=1}^N}\{\sum_{i=1}^N |f(x_i) - f(x_{i-1})| : 0 = x_0 < x_1 < \cdots < x_N = 1\}$.

Taking $f(x) = e(x)$, Lemma 4.1 gives that

$$\left|\sum_{1\leq i\neq j\leq N} e(X_i(x) - X_j(x))\right| \leq C\frac{\Delta_N}{N}$$

and with the choices $X_j(x) = a_j x$ we get that

$$\left|\sum_{j=1}^{N} e(a_j x)\right|^2 - N \leq C\frac{\Delta_N}{N}$$

and by the triangle inequality

$$\left|\sum_{j=1}^{N} \cos(2\pi a_j x)\right| \leq \left|\sum_{j=1}^{N} e(a_j x)\right|.$$

Let $f(k) = \sqrt{\log k}$; then we observe that

$$\sup_{k\geq 1} \frac{f(k^2)}{f(k)} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k[f(k)]^2} = \infty.$$

There exists a strictly increasing sequence of integers (a_k) such that

$$\limsup_{N\to+\infty} \frac{|\sum_{j=0}^{N} \cos(2\pi a_j x)|}{\sqrt{N} f(N)} > 0.$$

As shown in [BP] this can be done with a sequence satisfying

$$\frac{a_{k+1}}{a_k} \geq 1 + \frac{1}{\sqrt{k}(\log k)^{\beta}},$$

for some $\beta > 0$. In particular, we see that

$$\limsup_{N\to\infty} \frac{\Delta_N}{N \log N} > 0 \quad \text{a.e.}$$

In particular, Theorems 1, 2 and 3 cannot be improved much in general. ∎

5. A generalization

It is possible to extend these results from rectangles to certain types of more complicated sets. Assume that $(R_t)_{t=1}^{\infty}$ is a collection of disjoint k-dimensional rectangles in $\mathbb{T}^k$. Let us write

$$R_t = J_{t,1} \times \cdots \times J_{t,k},$$

where each J_{ti} is a half open interval, open on the left and closed on the right. We further assume that $\mathrm{leb}(R_t) = O(a^{-t})$, for some $a > 1$. Consider the set $B = \bigcup_{t=1}^{\infty} R_t$.

Theorem 5.1: *Given $\epsilon > 0$, there exists $N_0 = N_0(\epsilon, d, B, \epsilon)$ such that for $N > N_0$,*

$$\left| \frac{2}{N(N-1)} \sum_{1\leq i\neq j\leq N} \chi_B(X_i - X_j) - \mathrm{leb}(B) \right| = O\Big(\frac{(\log N)^{2+d+\epsilon}}{N}\Big), \quad \text{a.e.}$$

Proof: For each $N \in \mathbb{N}$, we can write $B = t(N) \cup s(N)$, where $t(M) = \bigcup_{1\leq t\leq z(N)} R_t$ and $t(M) = \bigcup_{t\geq z(N)} R_t$, where $z(N) = \log_a N$.

Let

$$K(B, l, x) = \sum_{1\leq i\neq j\leq l} \chi_B(X_i - X_j) - \frac{l(l-1)}{2} \mathrm{leb}(B).$$

If $||f||_2 = (\int_X |f|^2 dx)^{1/2|}$ we see that

$$||\max_{1\le l\le N} |K(t(N),l,x)|\,||_2 \le z(N)||\max_{1\le l\le N} \Delta_l||_2 \tag{5.1}$$

since

$$K(t(N),l,x) = \sum_{1\le i\ne j\le z(N)} \chi_B(X_i - X_j) - \frac{l(l-1)}{2}\operatorname{leb}(B).$$

Following the lines of the argument of the previous section we note that

$$||\max_{1\le l\le N} \Delta_l||_2 \le CN(\log N)^d.$$

We also have the bound

$$\begin{aligned} &||\max_{1\le l\le N} |K(s(N),l,x)|||_2 \\ &= \left\| \max_{1\le l\le N}\left|\frac{l(l-1)}{2}\sum_{1\le i\ne j\le l}\chi_{s(N)}(X_i - X_j) - \operatorname{leb}(s(N))\right\|_2 \right. \\ &\le \left\| \max_{1\le l\le N}\left|\frac{l(l-1)}{2}\sum_{1\le i\ne j\le l}\chi_{s(N)}(X_i - X_j)\right\|_2 + \operatorname{leb}(s(N))\right. \\ &\le \sum_{1\le i\ne j\le l} ||\chi_{s(N)}(X_i - X_j)||_2 + \frac{N(N-1)}{2}\operatorname{leb}(s(N)). \end{aligned}$$

Lemma 5.2: *Let* $R_{r,s,t} := \{\underline{x} : X_r - X_s \in R_t\}$. *Then* $\operatorname{leb}(R_t)$ *if* $r \ne s$ *and* $a_{ijr} \ne a_{ijs}$ *for all* i, j.

Proof: It is a simple exercise, repeatedly using the fact that for each integer $a \ne 0$ the map $x \mapsto \{ax\}$, and for any real y the map $x \mapsto \{x+y\}$, both preserve Lebesgue measure on the unit interval, that $|R_{r,s,t}| = |R_t|$ for each pair r, s with $r \ne s$. This means that

$$\begin{aligned} ||\chi_{s(N)}(X_i - X_j)||_2 &= \sum_{1\le i\ne j\le N}\left(\sum_{t>z(N)} |R_{i,j,t}|\right)^{1/2} + \frac{N(N-1)}{2}\operatorname{leb}(N) \\ &\le \sum_{1\le i\ne j\le N}\left(\sum_{t>z(N)} |R_t|\right)^{1/2} + \frac{N(N-1)}{2}\operatorname{leb}(N). \end{aligned}$$

Thus

$$||\max_{1\le l\le N} K(s(N),l,x)||_2 \le \frac{N(N-1)}{2}|s(N)|^{1/2}. \tag{5.2}$$

Furthermore,

$$(5.3)\qquad |s(N)| = \sum_{t>z(N)} \mathrm{leb}(R_k) << \sum_{t>z(N)} a^{-k} << a^{-z(N)}. \qquad \blacksquare$$

The above inequalities (5.1), (5.2) and (5.3) are sufficient to prove the theorem. $\blacksquare$

6. Discrepancy results

Using Lemma 1.1 directly instead of (1.1) the same methods lead to discrepancy results. Let us denote, for each $N \geq 1$,

$$D(N, \underline{x}) = D(X_1(\underline{x}), \dots, X_N(\underline{x})).$$

THEOREM 6.1: *Suppose $(X_N)_{N=1}^{\infty}$ is defined Type 1. Given $\epsilon > 0$, we have the bound*

$$D(N, \underline{x}) = O(N^{-1/2}(\log N)^{3/2+d+\epsilon}), \quad \text{a.e.}$$

THEOREM 6.2: *Suppose $(X_N)_{N=1}^{\infty}$ is Type 1. Given B and $\epsilon > 0$, then there exists $N_1 = N_1(\epsilon, d, B, \underline{x})$ such that for $N > N_1$ we have*

$$\left| \frac{1}{N} \sum_{j=1}^{N} \chi_B(X_j(\underline{x})) - \mathrm{leb}(B) \right| = O(N^{-1/2}(\log N)^{5/2+d+\epsilon}), \quad \text{a.e.}$$

THEOREM 6.3: *Suppose $(X_N)_{N=1}^{\infty}$ is Type 2. Given $\epsilon > 0$, we have the bound*

$$D(N, \underline{x}) = O(N^{-1/2}(\log N)^{1/2+d+\epsilon}), \quad \text{a.e.}$$

THEOREM 6.4: *Suppose $(X_N)_{N=1}^{\infty}$ is Type 2. Given B and $\epsilon > 0$, then there exists $N_1 = N_1(\epsilon, d, B, \underline{x})$ such that for $N > N_1$ we have*

$$\left| \frac{1}{N} \sum_{j=1}^{N} \chi_B(X_j(\underline{x})) - \mathrm{leb}(B) \right| = O(N^{-1/2}(\log N)^{3/2+d+\epsilon}), \quad \text{a.e.}$$

THEOREM 6.5: *Suppose $(X_N)_{N=1}^{\infty}$ is Type 4. Given $\epsilon > 0$, we have the bound*

$$D(N, \underline{x}) = O(N^{-1/4}(\log N)^{3/2+d+\epsilon}), \quad \text{a.e.}$$

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